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RTT presentation of finite $\mathcal{W}$-algebras

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Abstract

We construct a wide class of finite $\mathcal{W}$-algebras as truncations of Yangians. These truncations correspond to algebra homomorphisms and allow us to construct the $\mathcal{W}$-algebras as exchange algebras, the R-matrix being the Yangian one.

As an application, we classify all irreducible finite-dimensional representations of these $\mathcal{W}$-algebras and determine their centre.

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1. Introduction

It has already been proven [1] that there exists an algebra homomorphism between the Yangian based on $sl(N)$ and finite $\mathcal{W}(sl(Np), N \cdot sl(p))$-algebras. Such a connection plays a role in the study of physical models: for instance, in the case of the N-vectorial nonlinear Schrödinger equation on the real line, the full symmetry is the Yangian $Y(gl(N))$, but the space of states with particle number less than p is a representation of the $\mathcal{W}(gl(Np), p \cdot sl(N))$ algebra [2].

The connection between Yangians and finite $\mathcal{W}(sl(Np), N \cdot sl(p))$-algebras was proven in the Drinfeld presentation [3] of the Yangian. Since the homomorphism is (obviously) not an isomorphism, it does not allow us to carry the Yangian R-matrix 'down' to the finite $\mathcal{W}$-algebras. In this paper, we prove the correspondence in the 'RTT' presentation [4] of the Yangian. The mentioned finite $\mathcal{W}$-algebras[1] appear to be 'truncations' of the Yangian, i.e. the resulting coset when setting to zero the Yangian 'high-level' generators. These truncated Yangians have already been introduced in [5] under the name of Yangians of level p. Thanks to this presentation, we can deduce an R-matrix for the $\mathcal{W}$-algebras under consideration, as well as the complete classification of the finite-dimensional irreducible representations of these algebras. We also show that the Hopf structure of the Yangian cannot be carried by the homomorphism: although this is not a 'no-go theorem' for $\mathcal{W}$-algebras to be Hopf algebras, it severely constrains the possibilities of obtaining this structure.

[1] More precisely, it is the $\mathcal{W}(gl(Np), N \cdot sl(p))$-algebras with which we are concerned; we will return to this slight difference later on.

To prove our result, we need to combine three notions: Yangians, $\mathcal{W}$-algebras and cohomology. We have tried to be self-contained, and, as such, we need to recall known results for these different fields: this is done in section 2 for Yangians, in section 3 for $\mathcal{W}$-algebras and in appendix B for cohomology. We collect our results in section 4 and then present applications in section 5. We conclude with a ... conclusion, where possible generalizations and applications of our results are presented (section 6). Some calculations about $gl(Np)$ algebras are collected in appendix A.

2. Yangians

Yangians can be seen as deformations of loop algebras (based on a simple Lie algebra) and associated with a rational solution to the Yang–Baxter equation. They have been extensively studied, and we refer to [3, 6, 7] and references therein for more details. Here we will focus on Yangians based on $gl(N)$, and recall the basic properties below.

2.1. The Yangian $Y(gl(N))$

There are essentially two presentations of $Y(gl(N))$: one based on generators and relations [3] (Serre–Chevalley-type presentation), and the second (closer to integrable systems methods) using the R-matrix approach [4] (see also [6, 7] and references therein). Here we use the last one. The generators of the Yangian are gathered in a single matrix:

$$T(u) = \sum_{n=0}^{\infty} \sum_{i,j=1}^{N} u^{-n} T_n^{ij} E_{ij} = \sum_{n=0}^{\infty} u^{-n} T_n = \sum_{i,j=1}^{N} T^{ij}(u) E_{ij} \qquad \text{with} \quad T_0^{ij} = \delta^{ij} \tag{2.1}$$

where u is a spectral parameter and i, j indices in the fundamental of $gl(N)$. E_{ij} is the usual $N \times N$ matrix with unity at position (i, j). The algebraic structure is encoded in the relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \tag{2.2}$$

with $R(x) = 1 \otimes 1 - \frac{1}{x}P_{12}$ and P_{12} is the flip operator ($P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}$ in representations). The commutation relations read in components

$$[T_m^{ij}, T_n^{kl}] = \sum_{r=0}^{\min(m,n)-1} (T_r^{kj} T_{m+n-r-1}^{il} - T_{m+n-r-1}^{kj} T_r^{il}). \tag{2.3}$$

Note that in (2.3), all the couples (r, s), where $s = m+n-1-r$, satisfy $s < \min(m, n)$ and $r \geqslant \max(m, n)$.

It is known that the Yangian $Y(N)$ is a deformation of a loop algebra based on $gl(N)$. The parameter $\hbar$ can be recovered by multiplying the generators by an appropriate power of $\hbar$:

$$T_n^{ij} \rightarrow \hbar^{n-1} T_n^{ij}. \tag{2.4}$$

Then, the relations (2.3) can be rewritten as

$$[T_m^{ij}, T_n^{kl}] = \delta^{kj} T_{m+n-1}^{il} - \delta^{il} T_{m+n-1}^{kj} + o(\hbar) \tag{2.5}$$

which shows that $Y(N)$ is a deformation of a loop algebra (restricted to its positive modes). It can be proven that as soon as $\hbar \neq 0$, all the Hopf algebras $Y_\hbar(N)$ are isomorphic. The Hopf structure is given by

$$\Delta(T(u)) = T(u) \otimes T(u) \qquad \epsilon(T(u)) = 1 \quad S(T(u)) = -T(u) \tag{2.6}$$

or in components

$$\Delta(T_m^{ij}) = \sum_{k=1}^{N} \sum_{r=0}^{m} T_r^{ik} \otimes T_{m-r}^{kj} \qquad \epsilon(T_m^{ij}) = \delta_{m,0}\delta_{i,j} \quad S(T_m^{ij}) = -T_m^{ij}. \tag{2.7}$$

For brevity, we will denote $Y(N) \equiv Y(gl(N))$.

2.2. Centre of $Y(N)$ and associated Hopf subalgebras

The centre $\mathcal{D} = \mathcal{U}(d_i,\ i \in \mathbb{N})$ of $Y(N)$ is generated by the quantum determinant:

$$\text{q-det}(T(u)) \equiv \sum_{\sigma\in S_N} \text{sgn}(\sigma) T_{\sigma(1)1}(u) T_{\sigma(2)2}(u-1)\cdots T_{\sigma(N)N}(u-N+1) = 1 + \sum_{n=1}^{\infty} u^{-n} d_n. \tag{2.8}$$

The Hopf algebra $Y(sl(N))$ is the quotient of $Y(N)$ by the relation $\text{q-det}\,T = 1$, i.e. $Y(sl(N)) \sim Y(N)/\mathcal{D}$.

We introduce $\mathcal{D}_r = Y(N)\cdot\mathcal{U}(\{d_1, d_2, \ldots, d_r\})$. It is not difficult to show that for any value of r, $\mathcal{D}_r$ is a Hopf ideal of $Y(N)$. It is obviously an algebra ideal (because $\text{q-det}\,T(u)$ lies in the centre of the Yangian), and, from (2.7), one shows that

$$\Delta(\mathcal{D}_r) \subset \mathcal{D}_r \otimes \mathcal{D}_r \ \Rightarrow\ \Delta(\mathcal{D}_r) \subset \mathcal{D}_r \otimes Y(N) \oplus Y(N) \otimes \mathcal{D}_r \tag{2.9}$$

hence $\mathcal{D}_r$ is a coideal. Consequently, the coset $Y(N)/\mathcal{D}_r$ is also a Hopf algebra.

$$S_r Y(N) = Y(N)/\mathcal{D}_r \qquad \text{and} \qquad Y(N) \sim S_r Y(N) \otimes \mathcal{D}_r. \tag{2.10}$$

This allows us to construct a series of Hopf subalgebras:

$S_r Y(N) = Y(N)/\mathcal{D}_r$ and $Y(N) \sim S_r Y(N) \otimes \mathcal{D}_r \qquad \forall r$

$Y(N) \equiv S_0 Y(N) \supset S_1 Y(N) \supset \cdots \supset S_r Y(N) \cdots \supset S_\infty Y(N) \equiv Y(sl(N))$

where $Y(sl(N))$ is the only one which possesses a trivial centre. The intermediate subalgebras will be of some use in the following.

2.3. Evaluation representations

The finite-dimensional irreducible representations of $Y(N)$ have been classified [8, 9] (see also [6, 10] for more details). This uses the notion of evaluation representations [11, 12]:

Definition 2.1 (Evaluation representations). *An evaluation representation ev_π is a morphism from the Yangian $Y(gl(N))$ to a highest-weight irreducible representation π of $gl(N)$. The morphism is given by*

$$ev_\pi(T^{ij}_{(1)}) = \pi(T^{ij}_{(1)}) \qquad \textit{and} \qquad ev_\pi(T^{ij}_{(n)}) = 0 \qquad n > 1 \tag{2.11}$$

where we have identified the generators $T^{ij}_{(1)}$ with $gl(N)$ elements.

The evaluation representations form a very simple class of representations, since only one kind of Yangian generator is non-trivially represented. They are sufficient to obtain all finite-dimensional irreducible representations, through the tensor products of such representations:

Definition 2.2 (Tensor product of evaluation representations). *Let $\{ev_{\pi_i}\}_{i=1,\ldots,n}$ be a set of evaluation representations. The tensor product of these n representations $ev_{\vec{\pi}} = ev_{\pi_1} \otimes \cdots \otimes ev_{\pi_n}$ is a morphism from the Yangian $Y(gl(N))$ to the tensor product of $gl(N)$ representations $\vec{\pi} = \otimes_i \pi_i$ given by*

$$ev_{\vec{\pi}}(T^{ij}_{(r)}) = \bigoplus_{r_1+r_2+\cdots+r_n=r} \left(\bigotimes_{k=1}^{n} ev_{\pi_k}\big(T^{ij}_{(r_k)}\big) \right). \tag{2.12}$$

It satisfies

$$ev_{\vec{\pi}}(T^{ij}_{(r)}) \neq 0 \qquad \textit{if and only if} \quad r \leqslant n. \tag{2.13}$$

Note that this definition follows from the Yangian coproduct (2.6). Tensor products of evaluation representations play an important role in the classification of finite-dimensional irreducible representations of Yangians. This is reflected in the following theorems and corollary (proved in [8]; see also [9, 10, 13] for more details).

Theorem. *Any finite-dimensional irreducible representation of $Y(N)$ is highest weight and contains (up to multiplication by a scalar) a unique highest-weight vector.*

By highest-weight vector, we mean a vector η (in the representation) such that

$$t^{ij}(u)\eta = 0 \qquad 1 \leqslant i < j \leqslant N$$
$$t^{ii}(u)\eta = \lambda^i(u)\,\eta \qquad 1 \leqslant i \leqslant N$$

where $\lambda^i(u) = 1 + \sum_{r>0} \lambda^i_{(r)} u^{-r}$, with $\lambda^i_{(r)} \in \mathbb{C}$, and $t^{ij}(u)$ represents $T^{ij}(u)$. As usual, $\lambda(u) = (\lambda^1(u), \ldots, \lambda^N(u))$ is called the weight of the representation.

Theorem. *An irreducible highest-weight representation of $Y(N)$ of weight $\lambda(u)$ is finite dimensional if and only if there exist $(N-1)$ monic polynomials $P_i(u)$ such that*

$$\frac{\lambda^i(u)}{\lambda^{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)}.$$

In that case, the representation is isomorphic to the subquotient of the tensor product of $m = \sum_i m_i$ evaluation representations, where m_i is the degree of $P_i(u)$.

By monic polynomial, we mean a polynomial of the form

$$P_i(u) = \prod_{k=1}^{m_i} (u - \gamma_k) \qquad \text{with} \quad \gamma_k \in \mathbb{C}.$$

By subquotient, we mean the irreducible part of the highest-weight submodule of the mentioned tensor product. More precisely, in the tensor product of evaluation representations (which are by definition highest-weight representations), one considers the submodule generated by the tensor product of the highest-weight vectors, and quotients it by all (sub-) singular vectors which may appear.

Note that the tensor product is generically (but not always) irreducible (i.e. is equal to the mentioned submodule and has no singular vector). In the case of $Y(2)$, any irreducible finite-dimensional representation is isomorphic to a tensor product of evaluation representations (but there are still tensor products of evaluation representations which are reducible). Examples of these different situations are given in [10].

A simpler characterization of the finite-dimensional irreducible representations is given by the following corollary:

Corollary. *The irreducible finite-dimensional representations of $Y(N)$ are in one-to-one correspondence with the families $\{P_1(u), \ldots, P_{N-1}(u), \rho(u)\}$ where P_i are monic polynomials and $\rho(u) = 1 + \sum_{n>0} d_n u^{-n}$ encodes the values of the central elements.*

2.4. Truncated Yangians

The notion of truncated Yangians has been already introduced in [5] (although named not truncated, but Yangians of level p) as a tool in representation theory. They were also studied in [14]. We now introduce the left ideal generated by $\mathcal{T}_p = \mathcal{U}(\{T^{ij}_n, n > p\})$:

$$\mathcal{I}_p = Y(N) \cdot \mathcal{T}_p$$

and the coset (truncation of the Yangian at order p)

$$Y(N)_p = Y(N)/\mathcal{I}_p. \tag{2.14}$$

Property 2.3. *The truncated Yangian $Y(N)_p$ is an algebra ($\forall N \in \mathbb{N}$, $\forall p \in \mathbb{N}$). Δ is not a morphism of this algebra (for the structure induced by $Y(N)$).*

Proof. We prove the Lie algebra structure of $Y(N)_p$ by showing that $\mathcal{I}_p$ is a two-sided ideal, i.e. that we have $Y(N)\cdot\mathcal{I}_p \subset \mathcal{I}_p$ and $\mathcal{I}_p \cdot Y(N) \subset \mathcal{I}_p$. In fact, we will show a stronger property, that is

$$\big[Y(N)\,,\,\mathcal{T}_p\big] \subset Y(N)\cdot\mathcal{T}_p \qquad \text{and} \qquad \big[Y(N)\,,\,\mathcal{T}_p\big] \subset \mathcal{T}_p\cdot Y(N). \tag{2.15}$$

We make the calculation for the first inclusion, the proof for the other inclusion being identical. Indeed, the relation (2.3) shows that $[T_m^{ij}, T_n^{kl}]$ (for $n > p$) is the sum of two terms, the first being in $Y(N)\cdot\mathcal{T}_p$, the second belonging to $\mathcal{T}_p\cdot Y(N)$. Focusing on the latter, one rewrites it as

$$\begin{aligned}\sum_{r=0}^{\min-1} T^{kj}_{m+n-1-r}T^{il}_r &= \sum_{r=0}^{\min-1}\Big(T^{il}_r T^{kj}_{m+n-1-r} + \sum_{s=0}^{r-1}\big(T^{ij}_s T^{kl}_{m+n-2-s} - T^{ij}_{m+n-2-s}T^{kl}_s\big)\Big)\\ &= \sum_{r=0}^{\min-1} T^{il}_r T^{kj}_{m+n-1-r} + \sum_{s=0}^{\min-2}(\min - s - 1)\big(T^{ij}_s T^{kl}_{m+n-2-s} - T^{ij}_{m+n-2-s}T^{kl}_s\big)\end{aligned} \tag{2.16}$$

where min stands for $\min(m,n)$. In (2.16), only the last term belongs to $\mathcal{T}_p\cdot Y(N)$, with a summation which has one term less than the previous one: we can thus proceed recursively in a finite number of steps. The final result is an element of $Y(N)\cdot\mathcal{T}_p$.

As far as Hopf structure is concerned, the calculation

$$\Delta(T^{ij}_{p+1}) = T^{ij}_{p+1}\otimes 1 + 1\otimes T^{ij}_{p+1} + \sum_{n=1}^{p} T^{ik}_n \otimes T^{kj}_{p+1-n}$$

shows that $\mathcal{I}_p$ is not a coideal, since we have

$$\Delta(\mathcal{I}_p) \not\subset Y(N)\otimes\mathcal{I}_p \oplus \mathcal{I}_p\otimes Y(N).$$

Moreover, Δ is no longer an algebra morphism, since for instance

$$\Delta\big(\big[T^{ij}_p, T^{kl}_2\big]\big) - \big[\Delta(T^{ij}_p), \Delta(T^{kl}_2)\big] = \sum_{s+t=p}\big(T^{il}_{s+1}\otimes T^{kj}_t - T^{il}_s \otimes T^{kj}_{t+1}\big) \neq 0. \tag{2.17}$$

□

Finally, we note that each $Y(N)_p$ is a deformation of a truncated loop algebra based on $gl(N)$. By truncated loop algebra, we mean the quotient of a usual $gl(N)$ loop algebra (of generators t^{ij}_n) by the relations $t^{ij}_n = 0$ for $n < 0$ and $n > p$. The construction is the same as for the complete Yangian.

2.5. Poisson Yangians

In the following we will deal with a Poisson version of the Yangian, where the commutator is replaced by Poisson bracket (PB). It corresponds to the usual classical limit of quantum groups. One sets

$$T(u) = L(u) \qquad R_{12}(x) = \mathbb{I} - \hbar\, r_{12}(x) + o(\hbar) \qquad [\,,\,] = \hbar\{\,,\,\} + o(\hbar). \tag{2.18}$$

The relation (2.2) is then expanded as a series in $\hbar$, the first non-trivial term being the $\hbar^2$ coefficient. This new relation is the defining relation for the Poisson Yangian and reads

$$\left\{L(u) \overset{\otimes}{,} L(v)\right\} = [r_{12}(u-v), L(u) \otimes L(v)] \qquad \text{with} \quad r_{12}(x) = \frac{1}{x} P_{12} \tag{2.19}$$

where $\{L(u) \overset{\otimes}{,} L(v)\}$ is a matrix of component $\{L^{ij}, L^{kl}\}$ in the basis $E_{ij} \otimes E_{kl}$. In components

$$\{T^{ij}_m, T^{kl}_n\} = \sum_{r=0}^{\min(m,n)-1} (T^{kj}_r T^{il}_{m+n-r-1} - T^{kj}_{m+n-r-1} T^{il}_r). \tag{2.20}$$

Apart from the change from commutators to PB (and the commutativity of the product), all the above algebraic properties still apply.

In particular, we can still define the truncated (Poisson) Yangian, with the same procedure as above.

3. $\mathcal{W}$-algebras

Such algebras can be constructed by symplectic reduction of finite-dimensional Lie algebras in the same way as the conformal (affine) $\mathcal{W}$-algebras [15] arise as reduction of Kac–Moody (affine) Lie algebras [16], hence the name finite $\mathcal{W}$-algebras for the former [17]. Some properties of such $\mathcal{W}$-algebras have been developed [18–21]. In particular, starting from a simple Lie algebra $\mathcal{G}$, a large class of $\mathcal{W}$-algebras can be seen as the commutant, in a localization of the enveloping algebra $\mathcal{U}(\mathcal{G})$, of a $\mathcal{G}$-subalgebra [19]. This feature has already been exploited in various physical contexts [20, 21]. A remarkable fact is that the involved $\mathcal{W}$-algebras are just of the type $\mathcal{W}(sl(2n), n \cdot sl(2))$, a subclass of the $\mathcal{W}[gl(Np), N \cdot sl(p)]$ algebras, in which we are interested here.

We note $\mathcal{W}_p(N) \equiv \mathcal{W}[gl(Np), N \cdot sl(p)]$. This algebra is defined as the Hamiltonian reduction of the enveloping algebra of $gl(Np)$ (see below). In general, the $\mathcal{W}$-algebras are defined using semi-simple Lie algebras, but for $gl(m)$, we have the following property:

$$\mathcal{W}[gl(m), \mathcal{H}] \equiv \mathcal{W}[sl(m) \oplus gl(1), \mathcal{H}] \equiv \mathcal{W}[sl(m), \mathcal{H}] \oplus gl(1)$$

which allows us to extend the $\mathcal{W}$-algebra to $gl(m)$.

Note also that we are dealing with *finite* $\mathcal{W}$-algebra, i.e. the $gl(m)$-algebras we are speaking of are finite-dimensional Lie algebras (not their affinization).

We use the notations introduced in appendix A.

3.1. $\mathcal{W}_p(N)$ as an Hamiltonian reduction

Following the usual technique (see [17] and [21] for more details), we gather the generators of $gl(Np)$ in an $(Np) \times (Np)$ matrix:

$$\mathbb{J} = \sum_{a,b=1}^{N} \sum_{j=0}^{p-1} \sum_{m=-j}^{j} J^{ab}_{jm} M^{jm}_{ab} \tag{3.1}$$

where M^{jm}_{ab} are $(Np) \times (Np)$ matrices and J^{ab}_{jm} are in the dual algebra of $gl(Np)$. They obey PBs which mimic the commutation relations of $gl(Np)$:

$$\left\{J^{j,m}_{ab}, J^{\ell,n}_{cd}\right\} = \sum_{r=|j-\ell|}^{j+\ell} \sum_{s=-r}^{r} \left(\delta_{bc}\langle j, m; \ell, n|r, s\rangle J^{r,s}_{ad} - \delta_{ad}\langle \ell, n; j, m|r, s\rangle J^{r,s}_{cb}\right). \tag{3.2}$$

On the dual algebra, we introduce first-class constraints:

$$\mathbb{J}|_{\text{const}} = \epsilon_- + \sum_{a,b=1}^{N} \sum_{j=0}^{p-1} \sum_{m=0}^{j} J_{jm}^{ab} M_{ab}^{jm} \equiv \epsilon_- + \mathbb{B}. \tag{3.3}$$

Explicitly, these constraints are imposed on the negative grade generators $J_{jm}^{ab}, m < 0, \forall j, a, b$. They correspond to the vanishing of all these negative grade generators except $J_{1,-1}^{00}$, which is set to unity. We will denote them generically by ϕ_x. Physically, these first-class constraints generate gauge transformations, an infinitesimal form of which is

$$\delta_\lambda J_{jm}^{ab} \sim \sum_x \lambda_x \{\phi_x, J_{jm}^{ab}\} \tag{3.4}$$

where the symbol $\sim$ means that one has to impose the constraints once the PB has been computed. The interesting quantities are the gauge invariant ones, and it can be shown that a way to construct a basis for them is to choose a gauge fixing for $\mathbb{J}|_{\text{const}}$. In the present case, the gauge fixing is the highest-weight gauge:

$$\mathbb{J}|_{\text{g.f.}} = \epsilon_- + \sum_{a,b=1}^{N} \sum_{j=0}^{p-1} W_{jj}^{ab} M_{ab}^{jj} \equiv \epsilon_- + \mathbb{W} \tag{3.5}$$

where W_{jj}^{ab} are the (unknown) generators of the gauge invariant polynomials.

In other words, there is a unique set of parameters λ_x such that the gauge transformations (3.4) leads $\mathbb{J}|_{\text{const}}$ to $\mathbb{J}|_{\text{g.f.}}$. These parameters are polynomials in the original J_{jm}^{ab}, hence the generators W_{jj}^{ab}. Since they generate the gauge invariant polynomials, the W_{jj}^{ab} close (polynomially) under the PB: they generate the $\mathcal{W}(gl(Np), N \cdot sl(p))$-algebra. The Lie algebra structure of this $\mathcal{W}$-algebra is given by the PB (3.2), together with the knowledge of the polynomials W_{jj}^{ab}. Unfortunately, the complete expression of these polynomials is difficult to obtain in the general case, so that different techniques have been developed to compute the PB of the $\mathcal{W}$-algebra, without knowing the exact expression of the polynomials W_{jj}^{ab}.

There are essentially two different ways of defining the PB of the $\mathcal{W}_p(N)$-algebras: through the Dirac brackets, or using the so-called soldering procedure. We will need them both, and describe them in the following.

3.2. Dirac brackets

It can be shown that the first-class constraints together with the gauge fixing form a set of second-class constraints, i.e. that if $\Phi = \{\phi_\alpha\}_{\alpha\in I}$ is the set of all constraints, we have

$$\Delta_{\alpha\beta} = \{\phi_\alpha, \phi_\beta\} \text{ is invertible: } \sum_{\gamma\in I} \Delta_{\alpha\gamma} \bar{\Delta}^{\gamma\beta} = \delta_\alpha^\beta \qquad \text{where} \quad \bar{\Delta}^{\alpha\beta} \equiv (\Delta^{-1})_{\alpha\beta}. \tag{3.6}$$

Together with a set of second-class constraints occurs the notion of Dirac brackets which are constructed in such a way that they are compatible with these constraints:

$$\{X, Y\}_* \sim \{X, Y\} - \sum_{\alpha,\beta\in I} \{X, \phi_\alpha\} \Delta^{\alpha\beta} \{\phi_\beta, Y\} \qquad \forall X, Y \tag{3.7}$$

where the symbol $\sim$ means that one has to apply the constraints on the right-hand side *once the PBs have been computed.* The compatibility of the Dirac brackets with the constraints is reflected in the following property:

$$\{X, \phi_\alpha\}_* \sim 0 \qquad \forall\alpha \in I \quad \forall X. \tag{3.8}$$

Then, the PBs of the $\mathcal{W}$-algebra are defined as the Dirac brackets of the unconstrained generators J_{jj}^{ab}:

$$\{W_j^{ab}, W_\ell^{cd}\} \equiv \{J_{jj}^{ab}, J_{\ell\ell}^{cd}\}_*. \tag{3.9}$$

In the case we are considering, the matrix Δ takes the form

$$\begin{aligned}
\Delta^{ab;cd}_{jm;k\ell} &= \{J^{ab}_{jm}, J^{cd}_{k\ell}\} \qquad \forall a,b,c,d,j,k \quad \forall m<j \quad \forall \ell<k \\
&= (-1)^m \frac{j(j+1)-m(m+1)}{2}\frac{\eta_j}{\eta_1}\delta_{j,k}\delta_{m+\ell+1,0}\delta^{bc}\delta^{ad} \\
&\quad + \langle j,m;k,\ell|t,t\rangle \left(\delta^{bc}J^{ad}_{tt} - (-1)^{j+m+k+\ell}\delta^{ad}J^{cb}_{tt}\right) \\
&= (-1)^m \frac{j(j+1)-m(m+1)}{2}\frac{\eta_j}{\eta_1}\delta_{j,r}\delta_{m+s+1,0}\delta^{be}\delta^{af}\left(\mathbf{1}-\hat\Delta\right)^{cd;ef}_{k\ell;rs} \\
\hat\Delta^{ab;cd}_{jm;k\ell} &= \frac{2\eta_1\langle j,-m-1;k,\ell|t,t\rangle}{\eta_j(j(j+1)-m(m+1))}\left((-1)^m\delta^{ac}J^{bd}_{tt} + (-1)^{j+k+\ell}\delta^{bd}J^{ca}_{tt}\right).
\end{aligned} \tag{3.10}$$

The form (3.10) shows that Δ is invertible, for the matrix $\hat\Delta$ is nilpotent: due to the Clebsch–Gordan coefficient $\langle j,-m-1;k,\ell|t,t\rangle$, we have $(\hat\Delta)^{2p-1}=0$. Hence, we deduce

$$\bar\Delta^{jm;k\ell}_{ab;cd} = (-1)^{m+1}\frac{j(j+1)-m(m+1)}{2}\frac{\eta_1}{\eta_j}\sum_{n=0}^{2p-1}(\hat\Delta^n)^{ba;cd}_{j,-m-1;k\ell} \tag{3.11}$$

where we have set

$$(\hat\Delta^0)^{ab;cd}_{j,m;k\ell} = \delta_{j,k}\delta_{m,\ell}\delta^{ac}\delta^{bd}. \tag{3.12}$$

Once $(\Delta)^{-1}$ is known, one can compute the Dirac brackets. Unfortunately, in practice, (3.11) is difficult to achieve, and only partial results are obtained using the Dirac brackets.

3.3. Soldering procedure

The calculation of the PB of the $\mathcal{W}$-algebra can be achieved in another way, called the soldering procedure [16] (see also [1] in the case of finite $\mathcal{W}$-algebras). It is not our aim to show the equivalence of this approach with the previous (Dirac) procedure. Here we give just a flavour of it in the context of $\mathcal{W}_p(N)$-algebras.

In the soldering procedure, the idea is to view the (adjoint) action of the $\mathcal{W}_p(N)$-algebra on itself as a 'residual' action of the whole $gl(Np)$-algebra on the currents, residual in the sense that it 'respects' the constraints that have been imposed. In other words, among all the transformations induced by the (enveloping algebra of) $gl(Np)$, we look for those that do not affect the form $\mathbb{J}|_{\text{g.f.}}$: these will be the transformations induced by the $\mathcal{W}_p(N)$-algebra.

In this paper, thanks to the basis made explicit in appendix A, we will be able to synthetically present (and solve) this procedure in the case of $\mathcal{W}_p(N)$-algebras.

More precisely, the action of $gl(Np)$, with parameter $\lambda = \sum_{j,m;a,b}\lambda^{ab}_{jm}M^{jm}_{ab}$, can be written

$$\delta_\lambda\mathbb{J} = \{\mathrm{tr}(\lambda\mathbb{J}),\mathbb{J}\} = [\lambda,\mathbb{J}] \tag{3.13}$$

where {, } is the PB (on the J) and [,] is the commutator (of $Np\times Np$ matrices). Within all these transformations, we look for those which preserve the form of $\mathbb{J}|_{\text{g.f.}}$:

$$\delta_\lambda(\mathbb{J}|_{\text{g.f.}}) = \delta_\lambda\mathbb{W} = \sum_{j;a,b}(\delta_\lambda J^{ab}_{jj})M^{jj}_{ab}. \tag{3.14}$$

This constrains the parameters λ^{jm}_{ab}, and only N^2p of them are left free: they correspond to the parameters of the $\mathcal{W}$-transformation.

Explicitly, the calculation $[\lambda, \epsilon_- + \mathbb{W}] = \delta_\lambda \mathbb{W}$ leads to

$$\lambda^{j,m+1} = \sum_{k,r=0}^{p-1} \sum_{\ell=-k}^{k-1} \left(\lambda^{k\ell} W_r \langle k,\ell; r,r|j,m\rangle - W_r \lambda^{k\ell} \langle r,r; k,\ell|j,m\rangle\right) \quad \text{for} \quad -j \leqslant m < j \tag{3.15}$$

$$\delta_\lambda W_j = \sum_{k,r=0}^{p-1} \sum_{\ell=-k}^{k-1} \left(\lambda^{k\ell} W_r \langle k,\ell; r,r|j,j\rangle - W_r \lambda^{k\ell} \langle r,r; k,\ell|j,j\rangle\right) \tag{3.16}$$

where $\lambda_{j,m} = \sum_{a,b} \lambda_{jm}^{ab} M_{ab}^{jm}$, $W_j = \sum_{a,b} W_j^{ab} M_{ab}^{jj}$ and the products are matricial products.

The system (3.15) is strictly triangular in λ_{jm} with respect to the gradation $\mathrm{gr}(\lambda_{jm}) = j+m$. Indeed, the Clebsch–Gordan coefficients ensure that $|j-r| \leqslant k \leqslant j+r$ and $\ell + r = m$, so $\mathrm{gr}(\lambda_{k,\ell}) = k+\ell \leqslant j+m < j+m+1 = \mathrm{gr}(\lambda_{j,m+1}) =$ in (3.15). Thus, all the λ are expressible in terms of the $\lambda_{j,-j}$ parameters.

3.3.1. Calculation of $\{W_0^{ab}, W_j^{cd}\}$. As a start, we consider the variation of W_0. In this case, one has only to look at (3.16), which reads

$$\delta_\lambda W_0 = \sum_{k,r=0}^{p-1} \left(\lambda^{k,-k} W_r \langle k,-k; r,r|0,0\rangle - W_r \lambda^{k,-k} \langle r,r; k,-k|0,0\rangle\right) \tag{3.17}$$

$$= \sum_{k}^{p-1} (-1)^k \frac{\eta_k}{\eta_0} [\lambda^{k,-k}, W_r]. \tag{3.18}$$

Thus, we obtain the equation

$$\sum_j \tilde{\lambda}_j \{W_j, W_0\} = \frac{1}{p} \sum_j [\tilde{\lambda}_j, W_r] \tag{3.19}$$

where $\tilde{\lambda}_j = (-1)^j \eta_j \lambda_{j,-j}$. Hence, we are directly led to the PB:

$$\{W_0^{ab}, W_j^{cd}\} = \frac{1}{p} (\delta^{bc} W_j^{ad} - \delta^{ad} W_j^{cb}). \tag{3.20}$$

3.3.2. Calculation of $\{W_1^{ab}, W_j^{cd}\}$. Now, focusing on the variation of W_1 and using the results (A.23)–(A.28), we are led to

$$\delta_\lambda W_1 = \sum_j c_j \left(\frac{1}{j(2j-1)} \left[\lambda_{j-1,1-j}, W_j\right] - \left[\lambda_{j,1-j}, W_j\right]_+ - \frac{(j+1)(p-j-1)(p+j+1)}{2j+3} \left[\lambda_{j+1,1-j}, W_j\right] \right) \tag{3.21}$$

where c_j has been defined in (A.23) and [,] ([, $]_+$) stands for the commutator (anti-commutator) of $Np \times Np$ matrices. Then, solving the equation (3.15) for $m = -j, 1-j$, and plugging the result into (3.21), gives

$$\sum_j \tilde{\lambda}_j \{W_j, W_1\} = \frac{3}{p(p^2-1)} \left(\sum_{j=1}^{p-1} \frac{j(p^2-j^2)}{2j+1} [\tilde{\lambda}_{j-1}, W_j] + \sum_{j=1}^{p-1} \sum_{s=j}^{p-1} \left[\left[\tilde{\lambda}_s, W_{s-j}\right], W_j\right]_+ + \sum_{j=0}^{p-1} \sum_{s=j+1}^{p-1} \frac{s-j-1}{2j+1} \left[\left[\tilde{\lambda}_{s-1}, W_{s-j-1}\right]_+, W_j\right] - \sum_{j=0}^{p-1} \sum_{t=j+1}^{p-1} \sum_{s=t}^{p-1} \frac{1}{t(2j+1)} \left[\left[\left[\tilde{\lambda}_s, W_{s-t}\right], W_{t+1-j}\right], W_j\right] \right). \tag{3.22}$$

In components, we obtain the following PB:

$$
\begin{aligned}
\{W_1^{ab}, W_j^{cd}\} = \frac{3}{p(p^2-1)}\Bigg[& \frac{(j+1)(p^2-(j+1)^2)}{2j+3}\left(\delta^{cb}W_{j+1}^{ad} - \delta^{ad}W_{j+1}^{cb}\right) \\
& + j\left(\delta^{cb}(W_0W_j)^{ad} - \delta^{ad}(W_jW_0)^{cb} + W_j^{cb}W_0^{ad} - W_j^{ad}W_0^{cb}\right) \\
& + \sum_{s=1}^{j}\left(1+\frac{j-s}{2s+1}\right)\left(\delta^{cb}(W_sW_{j-s})^{ad} - \delta^{ad}(W_{j-s}W_s)^{cb}\right) \\
& + \sum_{s=1}^{j}\left(1-\frac{j-s}{2s+1}\right)\left(W_{j-s}^{ad}W_s^{cb} - W_s^{ad}W_{j-s}^{cb}\right) \\
& - \sum_{s=0}^{j-1}\sum_{t=s+1}^{j}\frac{1}{t(2s+1)}\Big(\delta^{cb}(W_sW_{t-s-1}W_{j-t})^{ad} - \delta^{ad}(W_{j-t}W_{t-s-1}W_s)^{cb} \\
& + W_{j-t}^{ad}(W_{t-s-1}W_s)^{cb} - (W_sW_{t-s-1})^{ad}W_{j-t}^{cb} \\
& + W_{t-s-1}^{ad}(W_{j-t}W_s)^{cb} - (W_sW_{j-t})^{ad}W_{t-s-1}^{cb} \\
& + W_s^{ad}(W_{j-t}W_{t-s-1})^{cb} - (W_{t-s-1}W_{j-t})^{ad}W_s^{cb}\Big)\Bigg]. \qquad (3.23)
\end{aligned}
$$

4. Comparison between truncated Yangians and finite $\mathcal{W}$-algebras

We have seen that the truncated Yangians are a deformation of a truncated loop algebra based on $gl(N)$. We show below that $\mathcal{W}_p(N)$ is also a deformation of this algebra, and that these two deformations coincide. We use here the notions presented in appendix B.

We work at the classical (PB) level.

4.1. $\mathcal{W}_p(N)$ as a deformation of a truncated loop algebra

To see that $\mathcal{W}_p(N)$ is a deformation of a truncated loop algebra based on $gl(N)$, we modify the constraints to

$$\mathbb{J} = \frac{1}{\hbar}\epsilon_- + \sum_{a,b=1}^{N}\sum_{j=0}^{p-1}\sum_{0\leqslant m\leqslant j} J_{jm}^{ab}M_{ab}^{jm}. \qquad (4.1)$$

These constraints are equivalent to the previous ones as soon as $\hbar \neq 0$. With these new constraints, the matrix Δ and its inverse read

$$
\begin{aligned}
(\Delta_\hbar)_{jm;k\ell}^{ab;cd} &= \frac{1}{\hbar}(-1)^m\frac{j(j+1)-m(m+1)}{2}\frac{\eta_j}{\eta_1}\delta_{j,k}\delta_{m+\ell+1,0}\delta^{bc}\delta^{ad}\left(1-\hbar\hat{\Delta}_{k\ell;rs}^{cd;ef}\right) \\
(\bar{\Delta}_\hbar)_{ab;cd}^{jm;k\ell} &= \hbar(-1)^{m+1}\frac{j(j+1)-m(m+1)}{2}\frac{\eta_1}{\eta_j}\sum_{n=0}^{2p-1}\hbar^n(\hat{\Delta}^n)_{j,-m-1;k\ell}^{ba;cd}.
\end{aligned} \qquad (4.2)
$$

Then, computing the Dirac brackets associated with these new constraints, one finds

$$\{J_{jj}^{ab}, J_{\ell\ell}^{cd}\}_\hbar = \{J_{jj}^{ab}, J_{\ell\ell}^{cd}\} - \sum_{efgh;kmrs}\{J_{jj}^{ab}, J_{km}^{ef}\}(\bar{\Delta}_\hbar)_{ef;gh}^{km;rs}\{J_{rs}^{gh}, J_{\ell\ell}^{cd}\} \qquad (4.3)$$

$$= \delta^{bc}J_{j+\ell,j+\ell}^{ad} - \delta^{ad}J_{j+\ell,j+\ell}^{cb} - \hbar P_\hbar(J) \qquad (4.4)$$

where $P_\hbar(J)$ (polynomial in the J_{jj}^{ab} which is computed using $\bar{\Delta}_\hbar$ as in section 3.2) has only positive (or null) powers of $\hbar$. This clearly shows that the $\mathcal{W}_p(N)$-algebra is a deformation of the algebra generated by $W_j^{ab} \equiv J_{jj}^{ab}$ and with defining (undeformed) PBs:

$$\{W_j^{ab}, W_\ell^{cd}\}_0 = \delta^{bc} W_{j+\ell}^{ad} - \delta^{ad} W_{j+\ell}^{cb} \qquad \text{if} \quad j+\ell < p \tag{4.5}$$

$$= 0 \qquad \text{if} \quad j+\ell \geqslant p. \tag{4.6}$$

One recognizes in this algebra an (enveloping) loop algebra based on $gl(N)$ quotiented by the relations $W_j^{ab} = 0$ if $j \geqslant p$. In other words, this algebra is nothing but a truncated loop algebra, and the $\mathcal{W}$-algebra is a deformation of it.

4.2. Identification of $\mathcal{W}_p(N)$ and $Y(N)_p$

We have already seen that the truncated Yangians as well as the $\mathcal{W}$-algebras we consider are both deformations of a truncated loop algebra:

$$\{W_j^{ab}, W_\ell^{cd}\}_1 = \{W_j^{ab}, W_\ell^{cd}\}_0 + \sum_{n=1}^{\infty} \hbar^n \varphi_n^W (W_j^{ab}, W_\ell^{cd}) \qquad 0 \leqslant j, \ell \leqslant p-1 \tag{4.7}$$

$$\{\bar{T}_m^{ij}, \bar{T}_n^{kl}\}_2 = \{\bar{T}_m^{ij}, \bar{T}_n^{kl}\}_0 + \sum_{r=1}^{\infty} \hbar^r \varphi_r^T (\bar{T}_m^{ij}, \bar{T}_n^{kl}) \qquad 0 \leqslant m, n \leqslant p-1 \tag{4.8}$$

where the cochains φ_n^W and φ_r^T obey (B.40) and (B.41). The undeformed PBs $\{,\}_0$ are identical (via the identification[2] $W_j^{ab} \equiv \bar{T}_{j-1}^{ab}$) and correspond to the truncated loop algebra. Thus, we have two deformed PBs $\{\,,\,\}_1$ and $\{\,,\,\}_2$, and all we need is to show that the cochains φ_n^W and φ_n^T coincide $\forall n$. To prove that this is indeed the case, we need the following properties:

Lemma 4.1. *Let $gl(N)_p$ be the loop algebra based on $gl(N)$, truncated at order p, and u_j^{ab} $(j < p)$ the corresponding generators. A two-cocycle φ with values in $\mathcal{U}(gl(N)_p)$ is completely determined once one knows $\varphi(u_0^{ab}, u_j^{cd})$ and $\varphi(u_1^{ab}, u_j^{cd})$, $\forall a, b, c, d = 1, \ldots, N$ and $\forall j = 0, \ldots, p-1$.*

Proof. We prove this lemma recursively. We write the cocycle condition for a triplet (u_1^A, u_j^B, u_k^C), using indices $A, B, C = 1, \ldots, N^2$ in the adjoint representation, and the commutation relations of $gl(N)_p$:

$$\begin{aligned} &f^{AB}{}_D\, \varphi(u_{1+j}^D, u_k^C) + f^{BC}{}_D\, \varphi(u_{k+j}^D, u_1^A) + f^{CA}{}_D\, \varphi(u_{1+k}^D, u_j^B) \\ &\quad = \{u_1^A, \varphi(u_j^B, u_k^C)\} + \{u_j^B, \varphi(u_k^C, u_1^A)\} + \{u_k^C, \varphi(u_1^A, u_j^B)\}. \end{aligned} \tag{4.9}$$

It can be rewritten as

$$\begin{aligned} \gamma_2\, \varphi(u_{1+j}^A, u_k^B) &= f_A{}^{CD} f_{DB}{}^E\, \varphi(u_1^C, u_{k+j}^E) + f_A{}^{CD} f_{DB}{}^E\, \varphi(u_j^C, u_{k+1}^E) \\ &\quad + f_A{}^{CD} \big(\{u_k^B, \varphi(u_j^C, u_1^D)\} + \{u_j^C, \varphi(u_1^D, u_k^B)\} + \{u_1^D, \varphi(u_k^B, u_j^C)\}\big) \end{aligned} \tag{4.10}$$

where $\gamma_2 \neq 0$ is the value of the second Casimir operator in the adjoint representation.

For $j = 1$, (4.10) allows us to compute $\varphi(u_2^D, u_k^C)$ $\forall C, D$ and $\forall k \geqslant 1$ once $\varphi(u_1^D, u_k^C)$ $\forall C, D$ and $\forall k$ is known.

Suppose now that we know $\varphi(u_j^A, u_k^B)$ for $1 \leqslant j < \ell_0$ and $\forall k$. Then, (4.10) for $j = \ell_0 - 1$ allows us to compute $\varphi(u_{\ell_0}^A, u_k^B)$ $\forall k$.

Thus, apart from the values $\varphi(u_0^A, u_k^B)$ we are able to compute all the expressions $\varphi(u_j^A, u_k^B)$. This ends the proof. □

[2] The shift $j \to j-1$ in the identification is due to a difference of convention between $\mathcal{W}$-algebras and Yangians: in the former case, the index j denotes the underlying $sl(2)$ representation, while in the latter j is the exponent of u in the formal series (2.1).

Property 4.2. *There exist two sets of generators* $\{{}^{\pm}\bar{W}_j^{ab}\}_{j=0,\ldots}$ *in* $\mathcal{W}_p(N)$ *such that*

$$\begin{aligned} &\{{}^{\pm}\bar{W}_1^{ab}\,,\ {}^{\pm}\bar{W}_j^{cd}\} = \delta^{cb}\,{}^{\pm}\bar{W}_{j+1}^{ad} - \delta^{ad}\,{}^{\pm}\bar{W}_{j+1}^{cb} + \bar{W}_0^{cb}\,{}^{\pm}\bar{W}_j^{ad} - {}^{\pm}\bar{W}_j^{cb}\,\bar{W}_0^{ad} \\ &\qquad \forall a,b,c,d = 1,\ldots,N \quad \forall j \geqslant 1 \\ &\{\bar{W}_0^{ab}\,,\ {}^{\pm}\bar{W}_j^{cd}\} = \delta^{cb}\,{}^{\pm}\bar{W}_j^{ad} - \delta^{ad}\,{}^{\pm}\bar{W}_j^{cb}. \end{aligned} \tag{4.11}$$

The generators ${}^{\pm}\bar{W}_j^{ab}$ *are polynomial of degree* $(j+1)$ *in the original ones* W_j^{ab} *and are recursively defined by*

$$\begin{aligned} &{}^{\pm}\bar{W}_{j,\pm}^{ab} = \sum_{n=1}^{j+1} {}^{\pm}\bar{W}_{j,(n)}^{ab} = \sum_{n=1}^{j+1}\sum_{|\vec{s}|=j+1-n} {}^{\pm}\alpha_{\vec{s}}^{n,j}\,(W_{s_1}\ldots W_{s_n})^{ab} \qquad 1<n \quad \text{and} \quad 1<j \\ &{}^{\pm}\bar{W}_1^{ab} = \pm\frac{p(p^2-1)}{6}\,W_1^{ab} + \frac{p(p\pm 1)}{2}\,(W_0W_0)^{ab} \\ &\bar{W}_0^{ab} \equiv {}^{+}\bar{W}_0^{ab} = {}^{-}\bar{W}_0^{ab} = p\,W_0^{ab} \end{aligned} \tag{4.12}$$

for some numbers ${}^{\pm}\alpha_{\vec{s}}^{n,j}$. *The summation on* $\vec{s}$ *is understood as a summation on n positive (or null) integers* $(s_1,\ldots,s_n)\equiv\vec{s}$ *such that* $|\vec{s}| = \sum_{i=1}^m s_i = j+1-n$.

The subsets $\{{}^{\pm}\bar{W}_j^{ab}\}_{j=0,\ldots,p-1}$ *form two bases of* $\mathcal{W}_p(N)$, *the other generators* $\{{}^{\pm}\bar{W}_j^{ab}\}_{j\geqslant p}$ *being polynomials in the basis elements.*

Proof. We first remark that the form (4.12) clearly shows that the p first generators are independent, and thus form a basis. The others must then be polynomials in any basis.

We prove the relation (4.11) by a recursion on j. It is easy to compute that the definitions are such that (4.11) is satisfied for $j=1$. For the recursion, we fix a basis ${}^{+}\bar{W}_j^{ab}$ or ${}^{-}\bar{W}_j^{ab}$ (the proof is obviously independent of the choice), and write it $\bar{W}_j^{ab}$.

We suppose that we have found generators $\bar{W}_j^{ab}$ for $j\leqslant j_0$ such that (4.11) is satisfied. This implies that we have

$$N\,\bar{W}_{j_0+1}^{cd} = \{\bar{W}_1^{ca}, \bar{W}_{j_0}^{ad}\} - \bar{W}_0^{aa}\,\bar{W}_{j_0}^{cd} + \bar{W}_0^{cd}\,\bar{W}_{j_0}^{aa} + \delta_{cd}\bar{W}_{j_0+1}^{aa}$$

where we have used implicit summation on repeated $gl(N)$ indices. Then, we obtain

$$\begin{aligned} N\{\bar{W}_1^{ab}, \bar{W}_{j_0+1}^{cd}\} &= \{\bar{W}_1^{ab}, \{\bar{W}_1^{ce}, \bar{W}_{j_0}^{ed}\}\} - \{\bar{W}_1^{ab}, \bar{W}_0^{ee}\,\bar{W}_{j_0}^{cd} - \bar{W}_0^{cd}\,\bar{W}_{j_0}^{ee}\} + \delta_{cd}\{\bar{W}_1^{ab}, \bar{W}_{j_0+1}^{aa}\} \\ &= \{\{\bar{W}_1^{ab}, \bar{W}_1^{ce}\}, \bar{W}_{j_0}^{ed}\} + \{\bar{W}_1^{ce}, \{\bar{W}_1^{ab}, \bar{W}_{j_0}^{ed}\}\} + \delta_{cd}\{\bar{W}_1^{ab}, \bar{W}_{j_0+1}^{aa}\} \\ &\quad -\bar{W}_0^{ee}\,\{\bar{W}_1^{ab}, \bar{W}_{j_0}^{cd}\} + \bar{W}_0^{cd}\,\{\bar{W}_1^{ab}, \bar{W}_{j_0}^{ee}\} + \bar{W}_{j_0}^{ee}\,\{\bar{W}_1^{ab}, \bar{W}_0^{cd}\} \\ &= \{\bar{W}_1^{cb}, \bar{W}_{j_0+1}^{ad}\} - \{\bar{W}_2^{cb}, \bar{W}_{j_0}^{ad}\} + \bar{W}_1^{ab}\,\bar{W}_{j_0}^{cd} - \bar{W}_1^{cd}\,\bar{W}_{j_0}^{ab} \\ &\quad +N\big(\bar{W}_0^{cb}\,\bar{W}_{j_0+1}^{ad} - \bar{W}_0^{ad}\,\bar{W}_{j_0+1}^{cb}\big) + \delta^{cb}\,A^{ad} - \delta^{ad}\,B^{cb} + \delta^{cd}\,C^{ab} \end{aligned}$$

with the notation

$$\begin{aligned} A^{ad} &= \{\bar{W}_2^{ae}, \bar{W}_{j_0}^{ed}\} + \bar{W}_0^{ad}\,\bar{W}_{j_0+1}^{ee} - \bar{W}_0^{ee}\,\bar{W}_{j_0+1}^{ad} + \bar{W}_1^{ad}\,\bar{W}_{j_0}^{ee} - \bar{W}_1^{ee}\,\bar{W}_{j_0}^{ad} \\ B^{cb} &= \{\bar{W}_1^{ce}, \bar{W}_{j_0+1}^{eb}\} + \bar{W}_0^{cb}\,\bar{W}_{j_0+1}^{ee} - \bar{W}_0^{ee}\,\bar{W}_{j_0+1}^{cb} \\ C^{ab} &= \{\bar{W}_1^{ab}\,,\ \bar{W}_{j_0}^{ee}\} + [\bar{W}_0, \bar{W}_{j_0+1}]^{ab}. \end{aligned}$$

It remains to compute $\{\bar{W}_2^{cb}, \bar{W}_{j_0}^{ad}\}$. This is done using the same techniques as above:

$$N\,\bar{W}_2^{cb} = \{\bar{W}_1^{ce}, \bar{W}_1^{eb}\} - \bar{W}_0^{ee}\,\bar{W}_1^{cb} + \bar{W}_0^{cb}\,\bar{W}_1^{ee} + \delta^{cb}\,\bar{W}_2^{ee}$$

so that we have

$$\begin{aligned} N\{\bar{W}_2^{cb}, \bar{W}_{j_0}^{ad}\} &= -\big(\{\bar{W}_1^{ab}, \bar{W}_{j_0+1}^{cd}\} + \{\bar{W}_1^{cd}, \bar{W}_{j_0+1}^{ab}\}\big) + N\big(\bar{W}_1^{ab}\,\bar{W}_{j_0}^{cd} - \bar{W}_1^{cd}\,\bar{W}_{j_0}^{ab}\big) \\ &\quad +N\big(\bar{W}_0^{ab}\,\bar{W}_{j_0+1}^{cd} - \bar{W}_0^{cd}\,\bar{W}_{j_0+1}^{ab}\big) + \delta^{ab}B^{cd} \\ &\quad +\delta^{cd}\big(-\{\bar{W}_{j_0+1}^{ae}\,,\ \bar{W}_1^{eb}\} + \bar{W}_0^{ee}\,\bar{W}_{j_0+1}^{ab} - \bar{W}_0^{ab}\,\bar{W}_{j_0+1}^{ee}\big) \\ &\quad +\delta^{cb}\big(\{\bar{W}_2^{ee}\,,\ \bar{W}_{j_0}^{ad}\} - [\bar{W}_0\,,\ \bar{W}_{j_0+1}]^{ad} - [\bar{W}_1\,,\ \bar{W}_{j_0}]^{ad}\big). \end{aligned} \tag{4.13}$$

Then, a recurrent use of these two brackets leads to the result

$$\begin{aligned}\{\bar{W}_1^{ab}\,,\ \bar{W}_{j_0+1}^{cd}\} &= \bar{W}_0^{cb}\,\bar{W}_{j_0+1}^{ad} - \bar{W}_0^{ad}\,\bar{W}_{j_0+1}^{cb} + \delta^{cb}\,\bar{W}_{j_0+2}^{ad} - \delta^{ad}\,\bar{W}_{j_0+2}^{cb} \\ &\quad - \frac{\delta^{ad}\delta^{cb}}{N(N^2-1)}\big(\{\bar{W}_1^{ee}\,,\ \bar{W}_{j_0+1}^{ff}\} - N\{\bar{W}_1^{ef}\,,\ \bar{W}_{j_0+1}^{fe}\}\big) \\ &\quad + \frac{\delta^{cd}}{N}\Bigg(\{\bar{W}_1^{ab}\,,\ \bar{W}_{j_0+1}^{ee}\} + [\bar{W}_0\,,\ \bar{W}_{j_0+1}]^{ab} \\ &\quad + \frac{\delta^{ab}}{N(N^2-1)}\big(\{\bar{W}_1^{ee}\,,\ \bar{W}_{j_0+1}^{ff}\} - N\{\bar{W}_1^{ef}\,,\ \bar{W}_{j_0+1}^{fe}\}\big)\Bigg)\end{aligned} \tag{4.14}$$

for some polynomials $\bar{W}_{j_0+2}^{ad}$.

Finally, we remark that the forms (4.12) and the PB (3.23) clearly show that the PB $\{\bar{W}_1^{ab}, \bar{W}_j^{cd}\}$ does not contain terms proportional to δ^{ab} or δ^{cd}. Moreover, a direct calculation, using (3.23), shows that

$$\{\bar{W}_1^{aa}, P_j^{bb}\} = 0 \qquad \forall\ P_j^{cd}(W) = \sum_{n=1}^{j+1}\ \sum_{|\vec{s}|=j+1-n} \beta_{\vec{s}}^{n,j}\,(W_{s_1}\ldots W_{s_n})^{cd}.$$

This is enough to show that the two last lines in the PB (4.14) identically vanish.

Hence, we can deduce that the PB must be of the form

$$\{\bar{W}_1^{ab}, \bar{W}_{j_0+1}^{cd}\} = \bar{W}_0^{cb}\,\bar{W}_{j_0+1}^{ad} - \bar{W}_0^{ad}\,\bar{W}_{j_0+1}^{cb} + \delta^{cb}\,\bar{W}_{j_0+2}^{ad} - \delta^{ad}\,\bar{W}_{j_0+2}^{cb} \tag{4.15}$$

which is exactly (4.11), so that the recursion on j is proven. □

We have computed the first and last terms ($\forall j \geqslant 0$) that appear in the definition (4.12):

$${}^{\pm}\bar{W}_{j,(1)}^{ab} = (\pm 1)^j\,(j!)^2\binom{p+j}{2j+1}\,W_j^{ab} \tag{4.16}$$

$${}^{-}\bar{W}_{j,(j+1)}^{ab} = \binom{p}{j+1}\,(\underbrace{W_0\cdots W_0}_{j+1})^{ab} \tag{4.17}$$

$${}^{+}\bar{W}_{j,(j+1)}^{ab} = \binom{p+j}{j+1}\,(\underbrace{W_0\cdots W_0}_{j+1})^{ab}. \tag{4.18}$$

Corollary 4.3. *The change of generators between $\{{}^{+}\bar{W}_j^{ab}\}$ and $\{{}^{-}\bar{W}_j^{ab}\}$ is given by*

$${}^{\pm}\bar{W}_j^{ab} = \sum_{n=1}^{j}(-1)^{j+n+1}\sum_{|\vec{s}|=j+1-n}({}^{\mp}\bar{W}_{s_1}\ldots{}^{\mp}\bar{W}_{s_n})^{ab}. \tag{4.19}$$

Proof. Using the expression (4.12) for $j = 1$ and the PB (4.11), one computes that

$$\big\{{}^{\pm}\bar{W}_1^{ab}\,,\ {}^{\mp}\bar{W}_j^{cd}\big\} = \delta^{bc}\big((\bar{W}_0{}^{\mp}\bar{W}_j)^{ad} - {}^{\mp}\bar{W}_{j+1}^{ad}\big) - \delta^{ad}\big(({}^{\mp}\bar{W}_j\,\bar{W}_0)^{cb} - {}^{\mp}\bar{W}_{j+1}^{cb}\big). \tag{4.20}$$

Then, a direct calculation shows that indeed the expression (4.19) satisfies (4.11). □

Corollary 4.4. *The basis $\{{}^{-}\bar{W}_j^{ab}\}$ is such that ${}^{-}\bar{W}_j^{ab} = 0$ for $j \geqslant p$. In the basis $\{{}^{+}\bar{W}_j^{ab}\}$, all the ${}^{+}\bar{W}_j^{ab}$ generators ($j \geqslant p$) are not vanishing.*

Proof. From the PB (4.11) it is clear that it is sufficient to show that $\bar{W}_p^{ab} = 0$. Writing this PB for $j = p$ and using the form

$$\bar{W}_p^{ab} = \sum_{n=2}^{p+1} \sum_{|\vec{s}|=p+1-n} \alpha_{\vec{s}}^{n,p} (\bar{W}_{s_1} \ldots \bar{W}_{s_n})^{ab} \tag{4.21}$$

one obtains only two possibilities for the α:

$$\alpha_{\vec{s}}^{n,p} = (-1)^n A \qquad \text{with} \quad A = 0 \text{ or } 1. \tag{4.22}$$

If $A = 0$ then $\bar{W}_p^{ab} = 0$, while if $A = 1$ the change of basis given in the corollary 4.3 shows that in the other basis we have $\bar{W}_p^{ab} = 0$. Hence, we have to determine which basis corresponds to $\bar{W}_p^{ab} = 0$.

Looking at the expressions (4.18) and (4.17), one concludes that ${}^-\bar{W}_p^{ab} = 0$ while ${}^+\bar{W}_j^{ab} \neq 0, \ \forall j$. □

In the following, we choose for $\mathcal{W}_p(N)$ the $\{{}^-\bar{W}_j^{ab}\}$ basis and omit the superscript—for the generators. Now, from above, it is easy to show:

Theorem 4.5. *The $\mathcal{W}_p(N)$-algebra is the truncated Yangian $Y(N)_p$.*

Proof. Let us first remark that the two algebras have identical (in fact undeformed) PBs on the couples $(\bar{W}_0^{ab}, \bar{W}_j^{cd})$, which proves that the cochains φ_n^W and φ_n^T coincide (in fact vanish) on these points.

Moreover, the property 4.2 shows that the cochains φ_n^W and φ_n^T coincide on the couples $(\bar{W}_1^{ab}, \bar{W}_j^{cd})$. Since φ_1 is a cocycle, this is enough (using lemma 4.1) to prove that φ_1^T and φ_1^W are identical.

Now, suppose that we have proven that φ_n^W and φ_n^T are identical for $n < n_0$. Then, equation (B.41) fixes $\varphi_{n_0}^W$ and $\varphi_{n_0}^T$, up to a cocycle:

$$\varphi_{n_0}^W = \varphi_{n_0} + \xi_{n_0}^W$$
$$\varphi_{n_0}^T = \varphi_{n_0} + \xi_{n_0}^T$$

where φ_{n_0} is a function of the cochains $\varphi_n^W = \varphi_n^T$, $n < n_0$. But property 4.2 shows that the two cocycles $\xi_{n_0}^W$ and $\xi_{n_0}^T$ coincide on the couples $(\bar{W}_1^{ab}, \bar{W}_j^{cd})$, which proves that they are identical (due to lemma 4.1).

Thus, $\varphi_{n_0}^W$ and $\varphi_{n_0}^T$ are identical, and we have proven recursively the property. □

4.2.1. Quantization. We have shown that truncated Yangians and $\mathcal{W}$-algebras coincide at the classical level. It remains to show that it is still true at the quantum level. Fortunately, an algebra morphism between Yangians and $\mathcal{W}$-algebras has already been given in [1], at classical and quantum levels. This relation was not sufficient to establish the identification between $\mathcal{W}$-algebras and truncated Yangians, since all the horizontal arrows involved in the diagram

$$\begin{array}{ccc} Y(N) & \longrightarrow & \mathcal{W}_p(N) \\ \updownarrow & & \updownarrow ? \\ Y(N) & \longrightarrow & Y_p(N) \end{array} \tag{4.23}$$

are not isomorphisms, hence the calculations performed in this paper.

However, once the relation (between $Y_p(N)$ and $\mathcal{W}_p(N)$) has been established at the classical level, we can use the result of [1] to promote it at the quantum level. More precisely,

$$
\begin{array}{ccccccc}
Y(N) & & & & & & S_\infty Y(N)_\infty \\
||| & \Rightarrow & S_1 Y(N)_\infty & \cdots \Rightarrow & S_p Y(N)_\infty & \cdots \Rightarrow & ||| \\
S_0 Y(N)_\infty & & & & & & Y(sl(N)) \\
\downarrow & & \downarrow & & \downarrow & \iddots & \\
\vdots & & \vdots & & \vdots & \iddots & \\
\downarrow & & \downarrow & & \downarrow & \iddots & \\
Y(N)_p & & S_1 Y(N)_p & & & & \\
||| & \Rightarrow & ||| & \cdots \Rightarrow & S_p Y(N)_p & & \\
\mathcal{W}_p(N) & & \mathcal{W}[sl(Np), N.sl(p)] & & & & \\
\downarrow & & \downarrow & & \iddots & & \\
\vdots & & \vdots & \iddots & & & \\
\downarrow & & \downarrow & \iddots & & & \\
Y(N)_1 & & S_1 Y(N)_1 & & & & \\
||| & \Rightarrow & ||| & & & & \\
\mathcal{U}(gl(N)) & & \mathcal{U}(sl(N)) & & & & \\
\downarrow & \iddots & & & & & \\
Y(N)_0 \equiv \{1\} & & & & & &
\end{array}
$$

Figure 1. The vertical links $\downarrow$ correspond to the truncations (algebra homomorphisms), while the horizontal links $\Rightarrow$ are associated with the coset by central elements (Hopf algebra homomorphisms).

now that we can identify the $\mathcal{W}_p(N)$-algebra with $Y_p(N)$ at the classical level, we can use the results of [1] at the quantum level: it has been established that any quantization of $\mathcal{W}_p(N)$ still obeys the Drinfeld relation, and hence the homomorphism still exists at the quantum level.

Thus, theorem 4.5 is valid at both classical and quantum levels, and figure 1 is correct (without a question mark).

Let us remark that in the proof we have established we have constructed $\mathcal{W}$-algebras as deformations of a truncated loop algebra and identified them with the truncated Yangians, i.e. truncations of deformed loop algebras. Denoting by $\mathcal{L}(gl(N))$ the loop algebra defined on $gl(N)$, and by $\mathcal{L}(gl(N))_p$ its truncation, the above sentence can be pictured as the following commutative diagram:

$$
\begin{array}{ccccc}
 & & Y(N) & & \\
 & \nearrow_\hbar & & \searrow^p & \\
\mathcal{L}(gl(N)) & & & & Y_p(N) \equiv \mathcal{W}_p(N) \\
 & \searrow^p & & \nearrow_\hbar & \\
 & & \mathcal{L}(gl(N))_p & &
\end{array}
\tag{4.24}
$$

where $\nearrow_\hbar$ represents a deformation, and $\searrow^p$ a truncation (at level p).

5. Applications

5.1. R-matrix for $\mathcal{W}$-algebras

The above construction allows us to associate the $\mathcal{W}$-algebras with the R-matrix of the Yangian, the difference between these two algebras lying in the mode development of $T(u)$: in both cases, the development is performed in powers of u^{-1}, but for the Yangian it is an infinite series, while the development is truncated to a polynomial for the $\mathcal{W}$-algebra. Explicitly, the presentation of the $\mathcal{W}_p(N)$-algebra takes the form

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \qquad \text{with} \qquad \begin{cases} T(u) = 1 + \sum_{n=1}^{p} \sum_{a,b=1}^{N^2} u^{-n} E_{ab}\, T_n^{ab} \\ R(x) = 1 \otimes 1 - \frac{1}{x} P_{12}. \end{cases}$$

Let us remark that this procedure is similar to the 'factorization procedure' which leads from the elliptic algebra $\mathcal{A}_{q,p}(N)$ to the Sklyanin algebra $S_{q,p}(N)$ [22] (see also [23] for more examples of factorizations). In all cases, one chooses for $T(u)$ a special dependence on u to obtain a finite algebra: this special dependence is nothing but a coset by some of the modes of $T(u)$. In all the examples, the Hopf structure of the starting algebra does not survive to this quotient.

Note that the R-matrix presentation of the $\mathcal{W}$-algebras provides an *exhaustive* set of commutation relations among the $\mathcal{W}_p(N)$ generators for generic N and p, while, up to now, a complete set of commutation relation was known only for a small number of $\mathcal{W}$-algebras.

Let us also remark that the R-matrix presentation allows us to define the $\mathcal{W}$-algebras without any reference to the underlying $gl(Np)$ algebra, and thus is a more 'abstract' definition.

5.2. Irreducible representations of $\mathcal{W}_p(N)$-algebras

Once again, the R-matrix presentation provides a very natural framework for the classification of $\mathcal{W}$-representations[3]. It is based on the notion of evaluation representations, as it appears in the Yangian context (see section 2.3). In fact, this classification was made in [5], in the context of (truncated) Yangians. We have the following theorem:

Theorem 5.1 (Finite-dimensional irreducible representations of $\mathcal{W}_p(N)$). *Any finite-dimensional irreducible representation of the $\mathcal{W}[gl(Np), N \cdot gl(p)]$ algebra is isomorphic to an evaluation representation or to the subquotient of the tensor product of at most p evaluation representations.*

Proof. By evaluation representations for $\mathcal{W}_p(N)$-algebra, we mean the definitions 2.1 and 2.2 with the change $T_r^{ab} \to W_{r-1}^{ab}$ (i.e. the evaluation representations of the truncated Yangian). The property (2.13) clearly shows that the (subquotient of the) tensor product of n evaluation representations is a representation of the truncated Yangian as soon as $n \leqslant p$. It also shows that if it is irreducible for the Yangian, then it is also irreducible for the truncated Yangian and that they are finite dimensional.

Now conversely, an irreducible representation π of the $\mathcal{W}_p(N)$-algebra can be lifted to a representation of the whole Yangian by setting $\pi(T_{(r)}^{ij}) = 0$ for $r > n$. It is then obviously irreducible for the Yangian, and thus is isomorphic to the tensor product of evaluation representations. □

We remark that the theorem 5.1 allows us to construct any (finite-dimensional) representation of $\mathcal{W}_p(N)$ in terms of p representations of $gl(N)$ (including trivial representations). This is exactly what one obtains from the so-called 'Miura transformation' that appears in the context of $\mathcal{W}$-algebras. Indeed, this transformation allows us to construct a representation of the $\mathcal{W}(\mathcal{G}, \mathcal{H})$-algebra using representations of $\mathcal{G}_0$, the zero-grade subalgebra of $\mathcal{G}$. In the case of $\mathcal{W}_p(N)$, we obtain $\mathcal{G}_0 = N \cdot gl(p)$, and hence need N representations of $gl(p)$, as stated in theorem 5.1.

Finally, as for Yangians, we have the following characterization (proved using the above theorem and the characterization for Yangians):

[3] We thank P Sorba for drawing our attention to this point.

Corollary 5.2. *The irreducible finite-dimensional representations of $\mathcal{W}_p(N)$ are in one-to-one correspondence with the families $\{P_1(u), \ldots, P_{N-1}(u), \rho(u)\}$ where P_i are monic polynomials of degree m_i such that $\sum_i m_i \leqslant p$, and $\rho(u) = 1 + \sum_{n=0}^{Np} d_n u^{-n}$.*

5.3. Generalization to $S_r Y(N)_p$ truncated Yangians

As well as defining truncated Yangians based on $Y(N)$, the same construction can be performed for each of the $S_r Y(N)$ Hopf algebras, to construct $S_r Y(N)_p$-algebras ($r \leqslant p$): these truncated Yangians will correspond to the quotient of $\mathcal{W}_p(N)$ by $\mathcal{D}_r$, which is a part of the $\mathcal{W}_p(N)$-centre (see below).

In particular, the $\mathcal{W}(sl(Np), N \cdot sl(p))$-algebra usually encountered in the literature is nothing but the truncation $S_1 Y(N)_p$. For this algebra, one sees that there exist two algebra homomorphisms: $Y(gl(N)) \to \mathcal{W}(sl(Np), N \cdot sl(p))$ and $Y(sl(N)) \to \mathcal{W}(sl(Np), N \cdot sl(p))$. The second one corresponds to the case given in [1].

More generally, we have the following property:

Property 5.3. *There is an algebra homomorphism from $S_r Y(N)_{p+q}$ to $S_{r+s} Y(N)_p$, for any values of $p, q, r, s = 0, 1, \ldots, \infty$.*

Proof. This is a trivial composition of algebra and Hopf algebra homomorphisms, as visualized in figure 1. □

5.3.1. Finite-dimensional irreducible representations of $S_r Y(N)_p$-algebras.

Starting from the theorem 5.1 and using cosets by central elements, it is easy to obtain:

Corollary 5.4. *Any finite-dimensional irreducible representation of the $S_r Y(N)_p$-algebra is obtained from the subquotient of tensor product of at most p evaluation representations, quotiented by r constraints on the generators of $\mathcal{D}_r$.*

The finite-dimensional irreducible representations of the $S_r Y(N)_p$-algebra are in one-to-one correspondence with the families $\{P_1(u), \ldots, P_{N-1}(u), \rho(u)\}$ where P_i are monic polynomials of degree m_i such that $\sum_i m_i \leqslant p$, and $\rho(u) = 1 + \sum_{n=0}^{Np-r} r d_n u^{-n}$.

In particular, in the case of the $\mathcal{W}(sl(Np), N \cdot sl(p))$-algebra, we obtain the result given in [1] for $N = 2$.

5.4. Centre of $\mathcal{W}_p(N)$-algebras

From the definition of $\mathcal{W}(gl(Np), N \cdot sl(p))$-algebras, one already knows that their centre contains the Casimir operators of $gl(Np)$, since, being central, these operators are obviously gauge invariant. Hence the dimension of the centre is at least Np. However, it was not proved (to our knowledge) that its dimension is exactly Np. Fortunately, the centre of the truncated Yangians $Y_p(N)$ has been determined in [5]:

Property 5.5. *A basis of the $Y_p(N)$ centre is given by all the coefficients of the principal part of the following generating function:*

$$H(x) = \sum_{w \in S_N} \sum_{i=1}^{N} \sum_{r_i=0}^{p-1} (-1)^{sg(w)} T_{r_1}^{w(1)1} T_{r_2}^{w(2)2} \cdots T_{r_N}^{w(N)N} \prod_{j=1}^{N} \left(\frac{(x-j)^{p-1-r_j}}{\prod_{k=1}^{p}(x-j-u_k)} \right) \qquad (5.25)$$

where S_N is the symmetric group and T_r^{ab} are the Yangian generators.

Looking at the poles of $H(x)$, it is easy to see that there are exactly Np poles (including multiplicities). A basis for this centre (using the quantum determinant) was also given in [14]. Hence, using this property and the above remark, we can deduce:

Corollary 5.6. *The centre of $\mathcal{W}_p(N)$ is Np dimensional and is given by $\mathcal{D}_{Np}/\mathcal{I}_p$. A basis of this centre is canonically associated with the Casimir operators of $gl(Np)$.*

Let us remark that the first p Casimir operators can be chosen as elements of the $\mathcal{W}_p(N)$-basis, while the next $p(N-1)$ are polynomials in the basis generators. Note that a different way to obtain these central generators has been given in [19]. It uses a determinant formula for $gl(Np)$ expressed for $\mathbb{J}_{gf}$, namely

$$\det(\mathbb{J}_{gf} - \lambda\,\mathbb{I}) = (-1)^{Np}\lambda^{Np} + \sum_{n=0}^{Np-1} C_{Np-n}\lambda^n. \tag{5.26}$$

More generally, the same reasoning leads to the following centre for $S_rY(N)_p$:

$$Z(S_rY(N)_p) = \mathcal{U}(d_{r+1}, \ldots, d_{pN})/\mathcal{T}_p. \tag{5.27}$$

It is generated by the last $(Np-r)$ independent Casimirs of $gl(Np)$.

6. Conclusion

We have shown that the finite $\mathcal{W}(gl(Np), N \cdot sl(p))$-algebras are nothing but truncated Yangians $Y(gl(N))_p$, i.e. cosets of the Yangian $Y(gl(N))$ by the relations $T^{ab}_{(n)} = 0$ for $n \geqslant p$. The resulting coset is an algebra, but the Yangian Hopf structure does not survive to the quotient. This property elucidates the algebra homomorphism between Yangians and finite $\mathcal{W}$-algebras, which was given in [1]. Using this property, we have been able to present these $\mathcal{W}$-algebras as exchange algebras, with the help of the Yangian R-matrix. This more abstract presentation is not linked to a Hamiltonian reduction, as usually defines the $\mathcal{W}$-algebras. It could be of some help in the search for a geometrical interpretation of $\mathcal{W}$-algebras. As a consequence, we have also given a complete classification of the finite-dimensional irreducible representations for these $\mathcal{W}$-algebras. This classification completes that given in [1] for $\mathcal{W}(sl(2n), 2 \cdot sl(n))$-algebras. Physically, one can hope to construct lattice models associated with $\mathcal{W}$-algebras, starting from models with Yangian symmetry.

Now that the relation between Yangians and $\mathcal{W}$-algebras is well understood, one can hope to construct R-matrices for general $\mathcal{W}$-algebras: work is in progress in this direction. Conversely, one can think of generalizing the notion of Yangians as certain limits of $\mathcal{W}(\mathcal{G}, \mathcal{H})$-algebras in which a (quasi-) Hopf structure can be recovered. This would provide a wide class of new types of quantum group.

Let us also remark that two other approaches for Yangians and $\mathcal{W}$-algebras could be related. On the one hand, one can construct Yangians as the projective limit of the centralizer of $gl(n)$ in $\mathcal{U}(gl(m+n))$ [24] (see also [25]), and on the other hand, some finite $\mathcal{W}$-algebras (of type $\mathcal{W}(gl(2n), n \cdot sl(2))$) have been realized as commutants of a $gl(2n)$ parabolic subalgebra in a certain localization of $\mathcal{U}(gl(2n))$ [19]. It seems to us quite natural to look for a global description of these two points of view.

Of course, the case of conformal $\mathcal{W}$-algebras (i.e. extensions of the Virasoro algebra) has to be considered. It could be related to a multi-parametric generalization of Yangians. Were such a generalization to be possible, one could think of an 'RTT presentation' of Virasoro algebra: this would allow us to relate 'usual' $\mathcal{W}$-algebras with the deformed $\mathcal{W}$-algebra presentation, a link which has not been clear up to now, since two different deformed algebras can be constructed [26, 27]. Note finally that the construction of some conformal $\mathcal{W}$-algebras (such as the Virasoro and the $\mathcal{W}_3$-algebras) as commutants in a localization of an affine Kac–Moody algebra (see the above paragraph) has been already achieved [19]: this could be a way to generalize the notion of Yangians, using the centralizer construction.

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Appendix A. General settings on $gl(Np)$

We have gathered here the notations and properties we need for the $gl(Np)$-algebra.

We consider the $gl(Np)$-algebra in its fundamental representation ($Np \times Np$ matrices), and take a basis adapted to the decomposition with respect to the $sl(2)$ algebra principal in $N \cdot sl(p) \equiv \underbrace{sl(p) \oplus \cdots \oplus sl(p)}_{N}$. This decomposition makes the 'factorization' $gl(Np) = gl(N) \otimes gl(p)$, valid in the fundamental representation, and the $sl(2)$ principal in $sl(p)$, naturally appear.

A.1. The principal embedding of $sl(2)$ in $sl(p)$

We will denote by $M_{j,m}$ (with $-j \leqslant m \leqslant j$ and $1 \leqslant j \leqslant p$) the $p \times p$ matrices resulting from the decomposition into $sl(2)$-multiplets:

$$[e_+, M_{j,m}] = \frac{j(j+1) - m(m+1)}{2} M_{j,m+1} \tag{A.1}$$

$$[e_-, M_{j,m}] = M_{j,m-1} \tag{A.2}$$

$$[e_0, M_{j,m}] = m\, M_{j,m} \tag{A.3}$$

$$[e_0, e_\pm] = \pm\, e_\pm \qquad \text{and} \qquad [e_+, e_-] = e_0 \tag{A.4}$$

where $e_{\pm,0}$ are the generators of the $sl(2)$-algebra principal in $sl(p)$. The normalizations in (A.1) and (A.2), although not symmetric, are adapted to the $\mathcal{W}$-algebra framework. When working with $gl(p)$ instead of $sl(p)$, we will add the $j = 0$ generator, proportional to the identity matrix.

The decomposition of $M_{j,m}$ in terms of the $p \times p$ matrices E_{ab} reads

$$M_{j,m} = \sum_{k=1}^{p-m} a^k_{j,m} E_{k,k+m} \qquad \text{with} \quad a^k_{j,m} = \sum_{i=0}^{j-m} (-1)^{i+j+m} \binom{j-m}{i} a^{k-i}_{j,j} \qquad \text{for} \quad 0 \leqslant m \leqslant j \tag{A.5}$$

$$M_{j,m} = \sum_{k=1}^{p+m} a^k_{j,m} E_{k-m,k} \qquad \text{with} \quad a^k_{j,m} = \sum_{i=0}^{j-m} (-1)^{i+j+m} \binom{j-m}{i} a^{k-i-m}_{j,j} \qquad \text{for} \quad -j \leqslant m \leqslant 0 \tag{A.6}$$

$$a^k_{j,j} = \frac{(k+j-1)!(p-k)!}{(k-1)!(p-k-j)!}. \tag{A.7}$$

The generators $e_{\pm,0}$ of the $sl(2)$-algebra are proportional to the $M_{1,m}$ generators:

$$\begin{aligned} e_+ &= \sum_{k=1}^{p-1} \frac{k(p-k)}{2} E_{k,k+1} = \frac{1}{2} M_{1,1} \\ e_0 &= \sum_{k=1}^{p} \left(\frac{p+1}{2} - k \right) E_{k,k} = -\frac{1}{2} M_{1,0} \\ e_- &= \sum_{k=1}^{p-1} E_{k+1,k} = -\tfrac{1}{2} M_{1,-1}. \end{aligned} \tag{A.8}$$

Let us remark that we have the following generating function for the coefficients ${}^{(p)}a^k_{j,m}$ (where (p) refers to the $gl(p)$ algebra under consideration):

$${}^{(p)}a^k_{j,m} = \frac{j!}{p!\,(k-1)!\,(j-m)!} \left[\frac{\mathrm{d}^p}{\mathrm{d}u^p} \frac{\mathrm{d}^j}{\mathrm{d}z^j} \frac{\mathrm{d}^{j-m}}{\mathrm{d}y^{j-m}} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\, a(x, y, z; u) \right]_{\substack{x=y=0 \\ z=u=0}} \tag{A.9}$$

$$\text{with} \quad a(x, y, z, u) = \frac{u}{\big(1 + y(1-x)\big)\big(1 - u[1 + z + x(1-u)]\big)}. \tag{A.10}$$

The scalar product is given by

$$\eta_{j,m;\ell,n} = (M_{j,m}, M_{\ell,n}) = \mathrm{tr}(M_{j,m} \cdot M_{\ell,n}) = (-1)^m \eta_j\, \delta_{j,\ell} \delta_{m+n,0} \tag{A.11}$$

$$\text{with} \quad \eta_j = (2j)!(j!)^2 \binom{p+j}{2j+1} \tag{A.12}$$

where the dot represents the matrix product and tr is the trace of matrices (in the p-dimensional representation).

In the following, we will need the Clebsch–Gordan-like coefficients given by

$$M_{j,m} \cdot M_{\ell,n} = \sum_{r=|j-\ell|}^{j+\ell} \sum_{s=-r}^{r} \langle j, m; \ell, n | r, s \rangle M_{r,s}. \tag{A.13}$$

As for usual Clebsch–Gordan coefficients, one can prove (using commutators by $e_{\pm,0}$) that r must be in $[|j-\ell|\,,\ j+\ell]$ and that s must be equal to $-m-n$. However, since we are in the fundamental of $sl(p)$, the coefficients will be truncated in such a way that only the values $r \leqslant p$ are kept in the decomposition (A.13). We will still call them Clebsch–Gordan coefficients.

Using the scalar product, one can compute these coefficients to be

$$\langle j, m; \ell, n | r, s \rangle = \frac{(-1)^s}{\eta_r}\, \mathrm{tr}\left(M_{j,m} \cdot M_{\ell,n} \cdot M_{r,-s} \right). \tag{A.14}$$

A.2. A few results concerning the Clebsch–Gordan-like coefficients

Using the cyclicity of the trace, one shows

$$\langle j, m; \ell, n | r, s \rangle = (-1)^{s+m} \frac{\eta_j}{\eta_r} \langle \ell, n; r, -s | j, -m \rangle = (-1)^{s+n} \frac{\eta_\ell}{\eta_r} \langle r, -s; j, m | \ell, -n \rangle. \tag{A.15}$$

We will also use the property

$$\langle j, m; \ell, n | r, s \rangle = \frac{(j-m)!(\ell-n)!(r+s)!}{(j+m)!(\ell+n)!(r-s)!} \langle \ell, -n; j, -m | r, -s \rangle \tag{A.16}$$

where the coefficients are due to the non-symmetric basis we have chosen.

With these two properties, one can compute

$$\langle r, -r; k, k|j, -j\rangle = (-1)^k \frac{\eta_{j+k}}{\eta_j} \delta_{r,j+k} \tag{A.17}$$

$$\langle k, k; r, -r|j, -j\rangle = (-1)^k \frac{\eta_{j+k}}{\eta_j} \delta_{r,j+k} \tag{A.18}$$

$$\langle r, 1-r; k, k|j, 1-j\rangle = (-1)^k \frac{j}{j+k} \frac{\eta_{j+k}}{\eta_j} \delta_{r,j+k} \tag{A.19}$$

$$\langle k, k; r, 1-r|j, 1-j\rangle = (-1)^k \frac{j}{j+k} \frac{\eta_{j+k}}{\eta_j} \delta_{r,j+k} \tag{A.20}$$

$$\langle r, -r; k, k|j, 1-j\rangle = (-1)^{k+1}\, k\, j\, \frac{\eta_{j+k-1}}{\eta_j} \delta_{r+1,j+k} \tag{A.21}$$

$$\langle k, k; r, -r|j, 1-j\rangle = (-1)^k\, k\, j\, \frac{\eta_{j+k-1}}{\eta_j} \delta_{r+1,j+k}. \tag{A.22}$$

We will also need the following coefficients:

$$\langle k, k; k, 1-k|1, 1\rangle = (-1)^{k+1} \frac{\eta_k}{\eta_1} \equiv c_k \tag{A.23}$$

$$\langle k, 1-k; k, k|1, 1\rangle = -c_k \tag{A.24}$$

$$\langle k, k; k-1, 1-k|1, 1\rangle = \frac{1}{k(2k-1)} c_k \tag{A.25}$$

$$\langle k-1, 1-k; k, k|1, 1\rangle = \frac{1}{k(2k-1)} c_k \tag{A.26}$$

$$\langle k+1, 1-k; k, k|1, 1\rangle = -\frac{(k+1)(p^2-(k+1)^2)}{2k+3} c_k \tag{A.27}$$

$$\langle k, k; k+1, 1-k|1, 1\rangle = -\frac{(k+1)(p^2-(k+1)^2)}{2k+3} c_k. \tag{A.28}$$

A.3. Basis for $gl(Np)$

We can use the above basis of $gl(p)$ to construct a basis for $gl(Np)$. Using the $N \times N$ matrices E_{ab}, the generators Υ_{ab}^{jm} of $gl(Np)$ in the fundamental will be represented by

$$\pi_F(\Upsilon_{ab}^{jm}) = M_{ab}^{jm} = E_{ab} \otimes M^{jm}. \tag{A.29}$$

The generators of the $sl(2)$-algebra principal in $N \cdot sl(p)$ are then

$$\epsilon_{\pm,0} = 1_N \otimes e_{\pm,0} \tag{A.30}$$

where $e_{\pm,0}$ are the $p \times p$ matrices defined above. We have the following commutation relations:

$$[\epsilon_+, M_{ab}^{j,m}] = \tfrac{1}{2}(j(j+1) - m(m+1)) M_{ab}^{j,m+1} \tag{A.31}$$

$$[\epsilon_-, M_{ab}^{j,m}] = M_{ab}^{j,m-1} \tag{A.32}$$

$$[\epsilon_0, M_{ab}^{j,m}] = m M_{ab}^{j,m} \tag{A.33}$$

$$[\epsilon_0, \epsilon_\pm] = \pm\epsilon_\pm \qquad \text{and} \qquad [\epsilon_+, \epsilon_-] = \epsilon_0 \tag{A.34}$$

together with

$$[M_{ab}^{00}, M_{cd}^{00}] = \delta_{bc} M_{ad}^{00} - \delta_{ad} M_{cb}^{00}. \tag{A.35}$$

This last commutator reveals the $gl(N)$-algebra which commutes with the $sl(2)$ subalgebra under consideration.

More generally, the product law (in the fundamental representation) reads

$$M_{ab}^{j,m} \cdot M_{cd}^{\ell,n} = \delta_{bc} \sum_{r=|j-\ell|}^{j+\ell} \sum_{s=-r}^{r} \langle j, m;\, \ell, n|r, s\rangle M_{ad}^{r,s} \tag{A.36}$$

which leads to the following commutation relations (valid in the abstract algebra):

$$[\Upsilon_{ab}^{j,m}, \Upsilon_{cd}^{\ell,n}] = \sum_{r=|j-\ell|}^{j+\ell} \sum_{s=-r}^{r} \left(\delta_{bc}\langle j, m;\, \ell, n|r, s\rangle \Upsilon_{ad}^{r,s} - \delta_{ad}\langle \ell, n;\, j, m|r, s\rangle \Upsilon_{cb}^{r,s}\right). \tag{A.37}$$

The scalar product is

$$\eta_{ab,cd}^{j,m;\ell,n} = (\Upsilon_{ab}^{j,m}, \Upsilon_{cd}^{p,n}) = \mathrm{tr}(M_{ab}^{j,m} \cdot M_{cd}^{p,n}) = \delta_{a,d}\,\delta_{b,c}\,\eta^{j,m;\ell,n}. \tag{A.38}$$

Appendix B. Deformations and cohomology

We include here some definitions (in the context of Chevalley cohomology) to be self-contained. For more details about deformations and their relation to cohomology, we refer to [28] and references therein.

B.1. A few words about Chevalley cohomology

We begin with an algebra $\mathcal{A}$, and first introduce the space $C_n(\mathcal{A}, \mathcal{A})$ of n-cochains with values in $\mathcal{A}$, i.e. skew-symmetric linear maps from $\wedge^n\mathcal{A}$ to $\mathcal{A}$. The Chevalley derivation δ maps n-cochains to $(n+1)$-cochains as

$$\begin{aligned}(\delta\chi_n)(u_0, \ldots, u_n) = {} & \sum_{i=0}^{n} (-1)^i \left\{u_i, \chi_n(u_0, u_1, \ldots, \hat{u_i}, \ldots, u_n)\right\} \\ & + \sum_{0\leqslant i<j\leqslant n} (-1)^{i+j}\, \chi_n\left(\{u_i, u_j\}, u_0, u_1, \ldots, \hat{u_i}, \ldots, \hat{u_j}, \ldots, u_n\right)\end{aligned}$$

where, as usual, $\hat{u_i}$ means that u_i has to be discarded in the list (or product, or sum, or whatever) we consider.

It can be shown that δ squares to zero:

$$(\delta(\delta\chi_n))(u_{-1}, u_0, u_1, \ldots, u_n) = 0 \qquad \forall u_{-1}, u_0, u_1, \ldots, u_n \quad \forall\chi_n \quad \forall n. \tag{B.39}$$

Thus, we introduce the cohomology associated with δ, i.e. we focus on Kerδ. Elements of Kerδ are called cocycles, and we will see that they play a direct role in the deformation of Lie algebras. The space of n-cocycles (with values in $\mathcal{A}$) is denoted $Z_n(\mathcal{A}, \mathcal{A})$: $\mathrm{Ker}\delta = \oplus_n Z_n(\mathcal{A}, \mathcal{A})$. Since $\delta^2 = 0$, we have $\mathrm{Im}\,\delta \subset \mathrm{Ker}\delta$: each n-cochain provides an $(n+1)$-cocycle. The elements $\delta\chi_n$ correspond to 'trivial' cocycles: they are called coboundaries, and the corresponding space denoted $B_n(\mathcal{A}, \mathcal{A})$. The cohomology describes the non-trivial cocycles, i.e. it is the space $H_n(\mathcal{A}, \mathcal{A}) = Z_n(\mathcal{A}, \mathcal{A})/B_n(\mathcal{A}, \mathcal{A})$, $H(\mathcal{A}, \mathcal{A}) = \oplus_n H_n(\mathcal{A}, \mathcal{A}) = \mathrm{Ker}\delta/\mathrm{Im}\,\delta$. Due to its definition, the Chevalley cohomology is naturally associated with Lie algebras. When the cochains take values in $\mathbb{C}$ instead of $\mathcal{A}$, the space $H_2(\mathcal{A}, \mathbb{C})$ classifies the non-trivial central extensions of $\mathcal{A}$: see for instance [29] where central extensions of generalized loop algebras are classified and computed. In the case we are considering, $C(\mathcal{A}, \mathcal{A})$ is related to deformations of $\mathcal{A}$.

B.2. Deformations

We start again with an algebra, with generators u_α ($\alpha \in \Gamma$).

$$\{u_\alpha, u_\beta\} = f^{\alpha\beta}{}_\gamma u_\gamma.$$

Actually, we will consider its enveloping algebra $\mathcal{A}$, and introduce a deformation of it

$$\{u_\alpha, u_\beta\}_\hbar = f^{\alpha\beta}{}_\gamma u_\gamma + \sum_{n=1}^{\infty} \hbar^n \varphi_n(u_\alpha, u_\beta)$$

where the antisymmetric bilinear forms φ_n take values in $\mathcal{A}$: they are all elements of $C_2(\mathcal{A}, \mathcal{A})$.

Asking the bracket $\{.,.\}_\hbar$ to obey the Jacobi identity leads to the following equations:

$$\delta\varphi_1 = 0 \tag{B.40}$$

$$\delta\varphi_n = \sum_{j+k=n} \big(\varphi_j(\varphi_k(u, v), w) + \varphi_j(\varphi_k(v, w), u) + \varphi_j(\varphi_k(w, u), v)\big) \qquad \text{for} \quad n > 1 \tag{B.41}$$

where the operation δ defined in section B.1 has naturally appeared.

These equations indicate that φ_1 is a cocycle, while φ_n is determined by the φ_p $(p < n)$ up to a cocycle. Note that $\delta\varphi_n$ is a coboundary, so the φ_p, $p < n$, must be such that the rhs of (B.41) is also a coboundary (it can be proven that this rhs is indeed a cocycle, i.e. is annihilated by δ). If the third cohomological space is not trivial, the rhs of (B.41) may be a cocycle while *not* being a coboundary: this leads to the usual assumption that the third cohomological space classifies the obstructions to deformations. In other words, it could appear that, in the attempt to construct a deformation, the chosen φ_p, $p < n$ are such that the lhs of (B.41) is a non-trivial cocycle, so one cannot solve this equation at level n. In that case, the deformation would be ill defined.

Fortunately, in the case we will consider below, we already know that we have well defined deformations, and we do not have to deal with a possible obstruction.

Note also that if φ_n is a coboundary

$$\varphi_n(u_\alpha, u_\beta) = \delta\chi_n(u_\alpha, u_\beta) = \{u_\alpha, \chi_n(u_\beta)\} - \{u_\beta, \chi_n(u_\alpha)\} - \chi_n(\{u_\alpha, u_\beta\})$$

we can perform a change of basis

$$\tilde{u}_\alpha = u_\alpha - \hbar^n \chi_n(u_\alpha)$$

such that, in this new basis, the term in $\hbar^n$ has disappeared:

$$\{\tilde{u}_\alpha, \tilde{u}_\beta\}_\hbar = f^{\alpha\beta}{}_\gamma \tilde{u}_\gamma + \sum_{m=1}^{n-1} \hbar^m \varphi_m(\tilde{u}_\alpha, \tilde{u}_\beta) + \sum_{m=n+1}^{\infty} \hbar^m \tilde{\varphi}_m(\tilde{u}_\alpha, \tilde{u}_\beta)$$

where $\tilde{\varphi}_m, m > n$ are new cochains resulting from the change of variables. In that sense, a coboundary leads to a trivial deformation. However, one has to be careful that to 'trivialize' the full deformation, the change of basis has to be done recursively and the coboundarity at level n has to be checked once the change of basis at level $n-1$ has been made (since the cochains are modified at higher order).

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